Chapter 3

Introduction to Proof Techniques

3.1 Introduction

These notes are primarily focused on the concept of mathematical proofs, rather than the specific mathematical subjects discussed. A common question that arises is whether one can learn how to create proofs. The answer is a clear no. The process of finding a proof can be extremely challenging, sometimes taking centuries. For instance, Aristotle conjectured that the diameter and circumference of a circle are incommensurable, implying that π is irrational, but it wasn't until 1761 that Lambert provided the first proof. Another famous example is Fermat's Last Theorem, which, despite being stated without a proof, waited more than 350 years before a proof was found.

While finding proofs can be difficult, reading and understanding them is typically easier. When reading a proof, the primary task is to verify the correctness of each step. This may be challenging due to the technical knowledge required or because the author may omit certain steps, expecting the reader to fill in the details. Occasionally, an unusual step in a proof might leave the reader questioning its purpose. In such cases, it is best to focus on the correctness of the step rather than its motivation. After understanding the proof, one can revisit the question of why that step was taken, although sometimes the original motivation may remain unclear, especially in proofs written long ago.

Learning proofs often involves memorizing the logical sequence of steps rather than the exact words or notation used. This is important because the types of arguments within a proof can be reused in different situations. A good strategy for mastering a proof is to read it a few times, then attempt to reconstruct it from memory. If stuck, one can refer back to the proof and continue this process until the proof can be fully recalled without assistance.

3.2 Predicate Logic and Proofs

3.2.1 Introduction to Predicate Logic

Predicate logic, also known as first-order logic, extends propositional logic by dealing with predicates and quantifiers, providing a more powerful framework for expressing mathematical statements. While propositional logic deals with simple, declarative statements that are either true or false, predicate logic allows us to express statements about objects and their properties.

3.2.2 Predicates and Quantifiers

A **predicate** is a statement that contains one or more variables and becomes a proposition when specific values are substituted for these variables. For example, the statement "x is greater than 2" can be written as a predicate $P(x)$, where $P(x)$ is true for values of $x > 2$ and false otherwise.

There are two main types of **quantifiers** used in predicate logic:

- **Universal Quantifier** ($\forall$): Expresses that a predicate is true for all elements in a certain set. For example, $\forall x \in \mathbb{R}, x^2 \geq 0$ states that for all real numbers x, x^2 is non-negative.
- **Existential Quantifier** ($\exists$): Expresses that there exists at least one element in a set for which the predicate is true. For example, $\exists x \in \mathbb{R}, x^2 = 1$ states that there exists a real number x such that $x^2 = 1$.

3.2.3 Logical Connectives

Predicate logic also uses logical connectives to form complex statements:

- **Conjunction** ($\wedge$): The statement $P(x) \wedge Q(x)$ is true if both $P(x)$ and $Q(x)$ are true. This can be also be refereed as "logical and".

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Disjunction** ($\vee$): The statement $P(x) \vee Q(x)$ is true if at least one of $P(x)$ or $Q(x)$ is true. This can be also be refereed as "logical or".

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Negation** ($\neg$): The statement $\neg P(x)$ is true if $P(x)$ is false. This can be also be refereed as "logical not".

P	$\neg P$
T	F
F	T

- **Implication** ($\rightarrow$): The statement $P(x) \rightarrow Q(x)$ is true if $P(x)$ being true implies that $Q(x)$ is also true.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Biconditional** ($\leftrightarrow$): The statement $P(x) \leftrightarrow Q(x)$ is true if $P(x)$ and $Q(x)$ are either both true or both false.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

3.2.4 Using Predicate Logic in Proofs

Predicate logic is fundamental in constructing mathematical proofs. It allows us to rigorously define the statements we are proving and provides the tools to manipulate these statements logically. Here are some examples of how predicate logic is used in proofs:

- **Proving Universal Statements**: To prove a statement of the form $\forall x, P(x)$, we must show that $P(x)$ holds for every possible x.
- **Proving Existential Statements**: To prove a statement of the form $\exists x, P(x)$, we must provide a specific x for which $P(x)$ is true.
- **Proof by Contradiction**: We assume the negation of the statement we wish to prove and show that this leads to a contradiction, thereby proving the original statement.

3.2.5 Examples

Example

Prove that the sum of any two even integers is even.
Proof: Proving universal statements. Let x and y be two even integers. By definition, there exist integers a and b such that $x = 2a$ and $y = 2b$. Consider the sum:

$$x + y = 2a + 2b = 2(a + b).$$

Since $a + b$ is an integer, $x + y$ is even.

Example

Prove that there exists an integer n such that $n^2 = n$.

Proof: Proving existential statements. Consider $n = 0$ and $n = 1$:

$$0^2 = 0 \quad \text{and} \quad 1^2 = 1.$$

Thus, $n = 0$ and $n = 1$ are integers that satisfy $n^2 = n$, so the statement is true.

3.3 What is a Mathematical Proof?

3.3.1 Definition of a Proof

A mathematical proof is a logical argument that demonstrates the truth of a mathematical statement P. The concept of proof has evolved over time. Early mathematics, such as in Babylonian and Egyptian cultures, was empirical. The formalization of proofs began with Euclid's *The Elements*, which introduced axiomatic systems. A proof consists of a sequence of statements $S_1, S_2, \ldots, S_n$ where:

1. Each S_i is either an axiom, an assumption, or derived from previous statements using rules of inference.
2. The final statement S_n is the statement P.

Historical Fact

Euclid's *The Elements*, written around 300 BCE, is one of the most influential works in the history of mathematics. It remained the main textbook for teaching mathematics, especially geometry, until the late 19th or early 20th century.

3.4 Proofs

3.4.1 Importance of Proofs

Proofs are essential in mathematics, ensuring that theorems are universally true. Unlike empirical sciences, where evidence supports a theory, mathematical truths are established through logical deduction.

Brainteaser

Can you prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$? Try to find a proof using induction!

3.4.2 Disproving Conjectures

Theorems are usually implicitly "for all" statements. For example, when we write "Let x and y be integers...", we mean "For all integers x and y...". Disproving a conjecture is straightforward: find one example where it isn't true. This is known as finding a counterexample.

Example

Disprove the conjecture: "If $\sqrt{x}$ is irrational, then x is irrational."
Solution: This conjecture is false for $x = 2$, as $\sqrt{2}$ is irrational, but 2 is rational.

3.4.3 Direct Proof

A direct proof is used to prove a conditional statement $p \to q$. We assume that p is true and follow logical implications to demonstrate that q must also be true.

Theorem Example

Theorem: If m is even and n is odd, then their sum $m + n$ is odd.
Proof: Since m is even, there exists an integer j such that $m = 2j$. Since n is odd, there exists an integer k such that $n = 2k + 1$. Therefore,

$$m + n = 2j + (2k + 1) = 2(j + k) + 1.$$

Since $j + k$ is an integer, $m + n$ is odd. ■

3.4.4 Proof by Contraposition

A proof by contraposition is used to prove a statement $p \to q$ by instead proving the equivalent statement $\neg q \to \neg p$.

Theorem Example

Theorem: If x is irrational, then $\frac{1}{x}$ is also irrational.
Proof: Suppose, for contraposition, that $\frac{1}{x}$ is rational. Then there exist integers m and n such that

$$\frac{1}{x} = \frac{m}{n}.$$

Thus, $x = \frac{n}{m}$, which is rational. This contradicts our assumption that x is irrational, so by contraposition, the theorem is true.

3.4.5 Proof by Contradiction

Proof by contradiction is a powerful technique based on the principle that if assuming the negation of a statement leads to a contradiction, then the statement itself must be true.

Theorem Example

Theorem: If n is an even perfect square, then n has an even square root.
Proof: Suppose, for contradiction, that $n = m^2$ for some odd integer m. Then $m = 2k + 1$ for some integer k. Thus,

$$n = m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is odd. This contradicts the premise that n is even. Therefore, m must be even.

3.4.6 Proof by Cases

In a proof by cases, the premise is divided into several cases, and each case is proven separately. If all cases are proven true, the entire theorem is proven true.

Theorem Example

Theorem: If n is an integer, then $3n^2 + n + 10$ is even.
Proof: We will prove by cases based on whether n is even or odd.
Case 1: Suppose n is even. Then there exists an integer k such that $n = 2k$.

$$3n^2 + n + 10 = 3(2k)^2 + 2k + 10 = 6k^2 + 2k + 10 = 2(3k^2 + k + 5),$$

which is even.
Case 2: Suppose n is odd. Then there exists an integer k such that $n = 2k + 1$.

$$3n^2 + n + 10 = 3(2k + 1)^2 + (2k + 1) + 10 = 12k^2 + 14k + 14 = 2(6k^2 + 7k + 7),$$

which is even.
Since both cases are proven, the theorem is true for all integers n.

3.4.7 Existence Proofs

Existence proofs are used to show that a particular object or number exists that satisfies a given condition.

Theorem Example

Theorem: There exists an integer n such that $|n^2 - 5| \leq 1$.
Proof: For $n = 2$, we have

$$|n^2 - 5| = |4 - 5| = 1,$$

which satisfies the condition.

3.5 Proof by Induction

One of the most important and intuitive proof techniques for me especially is Proof by Mathematical Induction, and we will address it here and solve some problems. Proof by induction is a powerful and widely used method for proving statements about integers. The basic idea behind induction is to prove that if a statement is true for some integer n, then it is also true for $n + 1$. By establishing that the statement is true for some initial value (usually $n = 0$ or $n = 1$), and then proving that its truth for one case implies its truth for the next, we can conclude that the statement is true for all integers greater than or equal to the initial value. Proof by induction is a versatile and essential tool in mathematics, particularly for proving statements about integers or recursively defined structures. By carefully following the base case and inductive step, one can rigorously establish the truth of a wide range of mathematical propositions.

3.5.1 Principle of Mathematical Induction

The principle of mathematical induction consists of two main steps:

1. **Base Case:** Prove that the statement is true for the initial value, usually $n = 0$ or $n = 1$. This step is essential because it provides the foundation for the induction.
2. **Inductive Step:** Assume that the statement is true for some arbitrary integer $n = k$. Then, prove that the statement is true for $n = k + 1$. This step involves showing that if the statement holds for k, it must also hold for $k + 1$.

If both steps are successfully completed, then by the principle of induction, the statement is true for all integers n greater than or equal to the base case.

3.5.2 Example of Proof by Induction

Let's consider an example to illustrate the method.

Theorem: For all integers $n \geq 1$, the sum of the first n positive integers is given by the formula:

$$S(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof:

Base Case: First, we verify the statement for $n = 1$.

$$S(1) = 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

So the statement holds for $n = 1$.

Inductive Step: Assume that the statement is true for some integer $n = k$. That is,

$$S(k) = \frac{k(k+1)}{2}$$

We need to show that the statement is true for $n = k + 1$, i.e.,

$$S(k+1) = 1 + 2 + 3 + \cdots + k + (k+1)$$

Using the inductive hypothesis:

$$S(k+1) = S(k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

We can factor out $(k + 1)$ from the right-hand side:

$$S(k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

This matches the form we wanted to prove, so the statement holds for $n = k + 1$.

Conclusion: By the principle of induction, the statement is true for all integers $n \geq 1$. Therefore, we have proven that

$$S(n) = \frac{n(n+1)}{2}$$

3.5.3 Geometric Example of Proof by Induction: Tiling a $2^n \times 2^n$rid Difficult

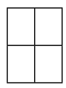
One classic example of a proof by induction involves tiling a grid of size $2^n \times 2^n$ with L-shaped tiles, each covering exactly 3 squares. The problem is to show that any $2^n \times 2^n$ grid with one square removed can be fully tiled with L-shaped tiles.

3.5.3.1 The Problem

Theorem: For any integer $n \geq 1$, a $2^n \times 2^n$ grid with one square removed can be completely covered by L-shaped tiles.

3.5.3.2 Proof by Induction

Base Case: Consider $n = 1$. A $2^1 \times 2^1 = 2 \times 2$ grid with one square removed looks like this:



This can be exactly covered by one L-shaped tile. Thus, the theorem holds for $n = 1$.

Inductive Step: Assume that the theorem holds for some $n = k$, i.e., any $2^k \times 2^k$ grid with one square removed can be tiled with L-shaped tiles. We need to prove that the theorem holds for $n = k + 1$.

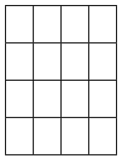
Consider a $2^{k+1} \times 2^{k+1}$ grid with one square removed. This grid can be divided into four $2^k \times 2^k$ sub-grids. There are two cases to consider:

1. If the removed square lies entirely within one of the four $2^k \times 2^k$ sub-grids, then by the inductive hypothesis, that sub-grid can be tiled with L-shaped tiles. The remaining three $2^k \times 2^k$ sub-grids can each have one square removed (from their corners) to place an L-shaped tile covering the corners of the three sub-grids. Thus, the entire $2^{k+1} \times 2^{k+1}$ grid can be tiled.
2. If the removed square lies on the boundary between two or more sub-grids, the idea is to place an L-shaped tile at the center of the grid, covering one square from each of three sub-grids, thereby reducing the problem to the previous case where the removed square is within a single sub-grid.

Conclusion: By the principle of mathematical induction, the theorem is true for all $n \geq 1$.

3.5.3.3 Visual Illustration

Here is a visual example for $n = 2$:



If we remove one square from this 4×4 grid, we can tile it using the process described above, with each step reducing the problem to tiling smaller sub-grids.

3.5.4 Variants of Induction

There are several variations of induction that are also useful:

3.5.4.1 Strong Induction

In strong induction, instead of assuming that the statement is true for a single integer k, we assume that it is true for all integers less than or equal to k. We then use this stronger assumption to prove that the statement holds for $k + 1$.

3.5.4.2 Structural Induction

Structural induction is a generalization of mathematical induction and is used to prove statements about objects that have a recursive structure, such as sequences, trees, or other mathematical structures defined inductively.

3.5.5 Common Pitfalls

- **Forgetting the Base Case:** The base case is crucial and cannot be omitted. Without it, the inductive step does not have a starting point.
- **Incorrect Inductive Step:** The inductive step must prove the statement for $n = k+1$ based on the assumption that it is true for $n = k$. Failing to correctly establish this can invalidate the proof.

- **Assuming What Needs to be Proven:** Induction requires that we assume the statement is true for $n = k$ only to prove it for $n = k + 1$. Assuming more than this can lead to circular reasoning.

3.6 Proof Writing Techniques

3.6.1 Tips for Clear and Effective Proof Writing

When writing mathematical proofs, clarity and precision are key. Here are some tips to help you write clear and effective proofs:

- **Structure your proof:** Begin by clearly stating what you are going to prove. Follow with the logical steps needed to arrive at the conclusion.
- **Use proper notation:** Ensure that your notation is consistent and follows standard mathematical conventions.
- **Be precise with language:** Avoid vague terms and be specific with your statements.
- **Explain each step:** Even if a step seems obvious, it is good practice to explain why it follows logically.
- **Review your proof:** After writing, review your proof to check for any errors or gaps in logic.

3.6.2 Common Errors and How to Avoid Them

Here are some common errors in proof writing and how you can avoid them:

- **Assuming what you need to prove:** Avoid circular reasoning by ensuring you do not assume the truth of the statement you are trying to prove.
- **Overlooking edge cases:** Consider all possible cases, including boundary or special cases that might require separate consideration.
- **Misusing notation:** Misinterpretation of mathematical symbols can lead to incorrect proofs. Be careful with your notation.
- **Skipping steps:** While brevity is important, skipping crucial logical steps can make your proof unclear or incomplete.

3.7 Further Reading and Resources

For those interested in further exploring proof techniques, especially for beginners preparing for the Mathematical Olympiad, the following resources are highly recommended:

- **Books:**
  - *The Art and Craft of Problem Solving* by Paul Zeitz
  - *Problem-Solving Strategies* by Arthur Engel
  - *Mathematical Olympiad Challenges* by Titu Andreescu and Răzvan Gelca
- **Papers:**
  - "Proof Techniques in Elementary Mathematics" by George Pólya
  - "The Role of Proof in Mathematics" by Michael Hardy
- **Online Resources:**
  - The Art of Problem Solving (AoPS) website, which offers a community and resources for students preparing for mathematics competitions.
  - The International Mathematical Olympiad (IMO) official website, which includes past problems and solutions.

3.8 Problems

1. Prove that the sum of any two odd integers is even.

2. Prove that the product of two rational numbers is rational.

3. Show that if n^2 is even, then n is even.

4. Prove that there exists an integer n such that n^2 is greater than 1000.

5. Use proof by contradiction to show that $\sqrt{2}$ is irrational.

6. Prove that the product of a nonzero rational number and an irrational number is irrational.

7. Prove by induction that for all integers $n \geq 1$, the sum of the first n cube of positive integers is given by:
$$S(n) = \left(\frac{n(n+1)}{2}\right)^2$$

8. Prove by induction that for all integers $n \geq 1$, the sum of the first n odd numbers is n^2.

9. Show that for any integer n, $n^2 \equiv 0 \text{ or } 1 \pmod 4$.

10. Prove that for any integer $n \geq 1$, $2^n > n^2$.

11. Disprove the conjecture: "If x^2 is rational, then x is rational."

12. Prove that there are infinitely many prime numbers.

13. Prove that the square root of a prime number is irrational.

14. Prove that if a divides b and b divides c, then a divides c.

15. Prove that for any two integers a and b, the greatest common divisor of a and b can be expressed as a linear combination of a and b.

16. Prove by induction that for all integers $n \geq 1$, $3^n - 1$ is divisible by 2.

17. Prove that for any integer $n \geq 1$, the sum of the squares of the first n positive integers is given by:
$$S(n) = \frac{n(n+1)(2n+1)}{6}$$

18. Show that for any integer n, $n^3 - n$ is divisible by 6.

19. Prove that if p is a prime number and p divides a^2, then p divides a.

20. Prove that the sequence $\{a_n\}$ defined by $a_1 = 2$ and $a_{n+1} = 3a_n + 1$ for $n \geq 1$ is strictly increasing.

21. Prove that the Fibonacci sequence $\{F_n\}$, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$, satisfies the identity:
$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

22. Show that for any positive integer n, the binomial coefficient $\binom{2n}{n}$ is even.

23. Prove that there are no positive integer solutions to the equation $x^2 + y^2 = 3z^2$.

24. Prove that for any real numbers $a, b, c \geq 0$, the following inequality holds:

$$(a + b + c)^2 \geq 3(ab + bc + ca)$$

25. Prove that for any integers a and b, if $a^2 + b^2$ is divisible by 3, then both a and b are divisible by 3.

26. Prove that for any prime number p, the factorial of $p-1$ is congruent to -1 modulo p:

$$(p - 1)! \equiv -1 \pmod{p}$$

27. Show that the equation $x^3 + y^3 + z^3 = 3xyz$ has infinitely many integer solutions.

28. Prove that for any positive integer n, the number $2^{2^n} + 1$ is composite for $n \geq 2$.

29. Prove that the sum of the angles in any convex polygon with n sides is $(n-2) \times 180°$.

30. Show that for any integer $n \geq 2$, the number $n! + 1$ is not a perfect square.

31. Prove that the sum of the reciprocals of the first n positive integers is not an integer for $n \geq 2$.

32. Let $x_1, x_2, \cdots, x_n$ be positive reals. Prove that

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

Chapter 4

Introduction to Combinatorics and Graph Theory

4.1 Introduction

Combinatorics and graph theory are two powerful and fundamental branches of mathematics that help us understand the underlying structures of complex systems. These fields are at the heart of problem-solving in diverse areas such as computer science, engineering, economics, social sciences, and biology. They provide tools for analyzing relationships within networks, counting distinct arrangements, optimizing processes, and understanding the ways in which discrete objects can be combined or connected.

Historical Fact

The concept of **combinatorics** dates back to the ancient Greeks, but it was not until the 17th century that it began to take shape as a distinct field of study. Blaise Pascal, renowned for his work in probability theory, is credited with laying the foundations of combinatorics through his famous *Pascal's Triangle*, which explores the relationships between numbers and their binomial coefficients. Even today, the triangle is used in various applications, from calculating probabilities to understanding patterns in nature.
Similarly, **graph theory** was born from a rather curious problem in 1736, when mathematician **Leonhard Euler** was tasked with finding a way to cross all seven bridges of the city of Königsberg (now Kaliningrad, Russia) without crossing the same bridge twice. Euler's solution to this problem led to the birth of graph theory. His work laid the groundwork for the study of networks and relationships, impacting everything from computer networks to social networks.

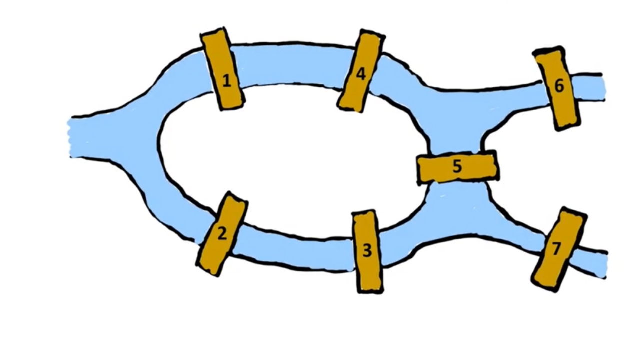


Fig. 4.1: The Seven Bridges of Königsberg.

In this chapter, we will explore both combinatorics and graph theory, starting with foundational concepts and progressing to more advanced topics. But before we dive into the details, let's start by asking: *How can counting and arranging objects help us solve real-world problems?* How can we understand complex networks, such as those that power our social media or manage our transportation systems?

At the heart of **combinatorics** lies the study of counting and arranging objects in specific ways. Whether you're planning a party and wondering how many seating arrangements are possible, or you're working on an optimization problem and need to calculate how resources can be allocated most efficiently, combinatorics has the answers. Key concepts like **permutations**, **combinations**, and **partitions** allow us to systematically calculate the number of possibilities for arranging or selecting items under various constraints. For example, permutations help answer questions like *"How many ways can I arrange 4 books on a shelf?"* while combinations answer *"How many ways can I choose 3 books from a shelf of 10, where order doesn't matter?"*

Graph theory has since found applications in a wide range of fields:

- **Social Networks:** Analyzing how individuals are connected to one another and studying network dynamics.
- **Transportation and Logistics:** Optimizing routes for delivery systems and minimizing travel times.
- **Computer Networks:** Understanding how data travels across the internet and ensuring efficient routing.

Whether you are designing efficient communication systems or simply planning the most effective way to connect points in a city, graph theory provides the tools to model and optimize these networks.

4.1.1 What You Will Learn In This Chapter?

1. Key combinatorial concepts such as **permutations**, **combinations**, and **partitions**.
2. The fundamental principles of **graph theory** and how graphs model networks.
3. Practical applications of these concepts in **real-world scenarios**, from social networks to secure communication.

As you proceed through this chapter, think of **combinatorics** as the art of organizing and counting the possibilities, and **graph theory** as the language of relationships and connections. Together, these tools allow mathematicians and scientists to solve complex problems by breaking them down into manageable parts, helping to better understand everything from social networks to the behavior of molecules.

Let's dive into the fascinating world of combinatorics and graph theory, where the art of counting and the study of networks meet to help solve some of the world's most intriguing problems.

4.1.2 Combinatorics: The Art of Counting and Combining

Combinatorics is a branch of mathematics that deals with counting and arranging objects in specific ways. While counting is a major part of combinatorics, the field extends to more advanced topics such as optimization, arrangements, selections, and partitioning of sets. In essence, combinatorics seeks to answer questions like:

- How many ways can a set of objects be arranged?
- How many ways can a subset of elements be selected?
- What are the possible configurations of a given set under certain constraints?

The key concepts in combinatorics include permutations, combinations, and partitions, all of which are fundamental in solving various real-world problems, such as calculating probabilities, optimizing resources, or designing systems.

4.1.3 Graph Theory: Studying Networks and Connections

Graph theory, another cornerstone of mathematics, deals with the study of graphs, which are mathematical models of networks. A graph is made up of vertices (also called nodes) connected by edges (also called links or arcs). These graphs are abstractions of various systems:

- Social networks, where vertices represent individuals and edges represent relationships.
- Computer networks, where vertices are devices, and edges represent communication links.
- Transportation systems, where vertices represent locations, and edges represent routes or connections.

Graph theory focuses on various properties of graphs, such as connectivity, flow, paths, cycles, and colorability, which help solve problems in network design, optimization, and system analysis.

Together, combinatorics and graph theory provide a comprehensive framework for solving problems involving discrete structures and interconnected systems. In this chapter, we will explore both of these fundamental areas, starting with basic counting principles and extending to more complex concepts.

4.2 Combinatorics: Key Concepts

Combinatorics is concerned with counting, arranging, and analyzing discrete structures. This section introduces foundational combinatorial concepts, such as permutations, combinations, and partitions, which are used to solve problems across a variety of fields.

4.2.1 Permutations: Arrangements of Objects

Definition

A **permutation** of a set is an arrangement of its elements in a specific order. The number of permutations of n distinct elements is denoted by $n!$ (read as "n factorial") and is calculated as:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1.$$

Factorials grow rapidly as n increases. The permutation concept applies to problems where the arrangement of elements matters. For example, arranging letters of a word or assigning seats in a row involves permutations.

Example

Consider the word "MATH". How many different ways can the letters in the word be arranged? Solution: There are 4 distinct letters in the word, so the number of permutations is:

$$4! = 4 \times 3 \times 2 \times 1 = 24.$$

Thus, there are 24 possible arrangements of the word "MATH".

Permutations can also be extended to cases where some elements are repeated. For example, if you want to count the distinct permutations of the word "SUCCESS," you would divide by the factorials of the repeated elements, such as:

$$\frac{7!}{2!3!} = 420.$$

4.2.2 Combinations: Selections Without Regard to Order

Definition

A **combination** of a set is a selection of its elements without regard to the order of selection. The number of ways to choose k elements from a set of n elements is given by the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This counts the number of ways to select k objects from a set of n objects, disregarding the order of selection.

Combinations are fundamental in problems where the order of the selected items doesn't matter. A good example is choosing team members from a group, or selecting lottery numbers.

Example

How many ways can you choose 2 letters from the set $\{A, B, C, D\}$? Solution: The number of ways to choose 2 letters from 4 is:

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \times 3}{2 \times 1} = 6.$$

Thus, there are 6 possible combinations: $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$.

Combinations appear frequently in probability theory and statistics, particularly when dealing with random sampling or drawing subsets from a population.

4.2.3 Partitions: Decomposing Numbers or Sets

Definition

A **partition** of a positive integer n is a way of writing n as a sum of positive integers, where the order of the addends does not matter. For example, the partitions of 4 are:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Partitions are used in number theory, combinatorics, and various optimization problems. In problems involving grouping or distributing items, partitioning is a key technique.

4.3 The Pigeonhole Principle

The Pigeonhole Principle is a simple yet powerful concept in combinatorics that can be used to prove the existence of certain patterns or structures in various situations. It is based on the intuitive idea that if you place more objects into fewer containers than there are objects, at least one container must hold more than one object. Despite its simplicity, the Pigeonhole Principle has numerous applications in number theory, computer science, and combinatorics.

4.3.1 The Basic Pigeonhole Principle

Definition

The basic form of the Pigeonhole Principle states: If n objects are placed into m containers and $n > m$, then at least one container must hold more than one object.

This principle is a straightforward application of the idea of counting. It tells us that when objects are distributed among containers, if there are more objects than containers, at least one container must contain more than one object. It is called the "Pigeonhole Principle" because it is often framed in terms of pigeons and pigeonholes: if you have more pigeons than pigeonholes, some pigeonholes must contain more than one pigeon.

Below is an image that visually represents the Pigeonhole Principle, where pigeons are placed into holes. It illustrates the idea that when the number of pigeons exceeds the number of holes, at least one hole must contain more than one pigeon.

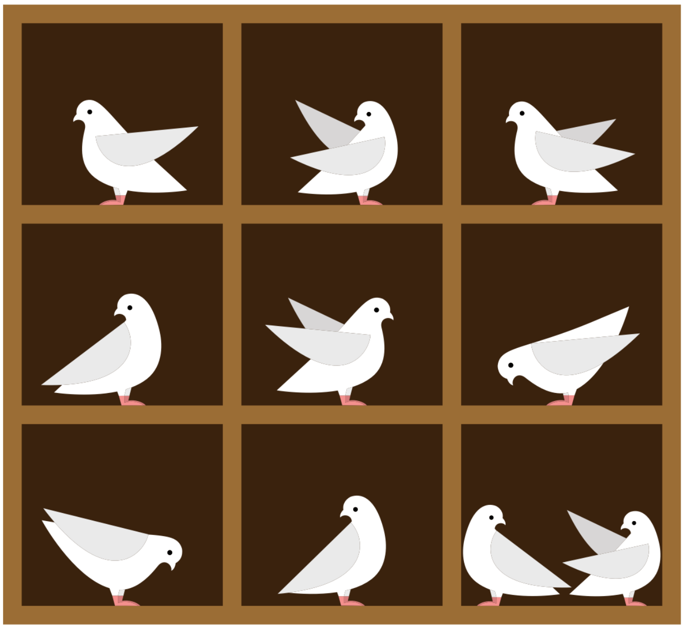


Fig. 4.2: Illustration of the Pigeonhole Principle.

Historical Fact

The Pigeonhole Principle was first formally stated by the mathematician Peter Gustav Lejeune Dirichlet in 1834, and it has since become a fundamental idea in combinatorics and number theory.

4.3.2 Examples of the Pigeonhole Principle

4.3.2.1 Simple Example

Suppose you have 10 people, and you want to assign them to 9 rooms. According to the Pigeonhole Principle, at least one room must contain more than one person because there are more people than rooms. This is a simple application of the principle, showing that when the number of objects exceeds the number of containers, some container will hold more than one object.

4.3.2.2 Example with Divisibility

A more interesting example involves divisibility. If you have 6 integers and want to place them into 3 groups based on their remainders when divided by 3, the Pigeonhole Principle guarantees that at least two of the integers will have the same remainder. This is because there are only 3 possible remainders (0, 1, and 2), so with 6 integers, two must share a remainder.

4.3.3 Generalization: The Pigeonhole Principle for Multiple Containers

The Pigeonhole Principle can also be generalized. If n objects are placed into m containers, and $n > k \times m$ for some integer k, then at least one container must contain at least $k + 1$ objects. This generalized form can be useful in more complex problems where you need to ensure a certain number of objects are placed in a container.

4.3.4 Applications of the Pigeonhole Principle

The Pigeonhole Principle has various applications in mathematics and computer science, including:

- **Number Theory:** It is used in proofs involving divisibility, modular arithmetic, and finding pairs of numbers with specific properties.
- **Graph Theory:** In graph coloring problems, the Pigeonhole Principle can be used to prove that certain colorings are impossible or that some vertices must share a color.
- **Cryptography:** It is applied in hash functions and proofs related to the distribution of data.
- **Scheduling Problems:** The Pigeonhole Principle is used to show that scheduling certain tasks or events will inevitably lead to conflicts.

4.3.5 Brain Teasers Involving the Pigeonhole Principle

Here are some fun problems related to the Pigeonhole Principle to test your understanding:

1. **Pigeonhole with Socks:** In a drawer, there are 10 red socks, 12 green socks, and 8 blue socks. How many socks must you take out to guarantee that you have at least one matching pair of socks?
2. **Birthday Paradox:** How many people need to be in a room for the probability of at least two of them sharing a birthday to be greater than 50
3. **Divisibility Pigeonhole:** How many integers between 1 and 100 (inclusive) must be chosen to guarantee that at least two of them are divisible by 3?

4.4 The Binomial Theorem and Its Applications in Counting

The Binomial Theorem is a fundamental result in algebra that describes the expansion of powers of binomials. It plays a crucial role in combinatorics, particularly in problems involving counting subsets, arrangements, and selections. The theorem provides a way to expand expressions of the form $(x+y)^n$, where n is a non-negative integer, into a sum of terms involving binomial coefficients.

4.4.1 The Binomial Theorem

Definition

The Binomial Theorem states that for any integer $n \geq 0$, we have:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

where $\binom{n}{k}$ is the binomial coefficient, which counts the number of ways to choose k elements from a set of n elements, given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The binomial coefficients $\binom{n}{k}$ are central to combinatorics, as they appear in many counting problems. For instance, $\binom{n}{k}$ counts the number of ways to choose k objects from a set of n objects, regardless of order.

4.4.2 Applications of the Binomial Theorem in Counting

4.4.2.1 Counting Subsets

One of the most common applications of the Binomial Theorem is in counting subsets. For any set with n elements, the total number of subsets of the set (including the empty set and the set itself) is 2^n. This is equivalent to expanding the binomial $(1+1)^n$, where each element can either be included or excluded from a subset. The Binomial Theorem gives:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}.$$

Thus, the total number of subsets is:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

This result shows that the number of subsets of a set of size n is equal to the sum of the binomial coefficients $\binom{n}{k}$ for all k from 0 to n.

4.4.2.2 Counting Ways to Choose Objects

The Binomial Theorem is also useful in problems where you need to count the number of ways to select k objects from a set of n objects. For example, in a committee selection problem where you need to choose k people from a group of n, the number of possible ways to make the selection is given by the binomial coefficient:

$$\binom{n}{k}.$$

This is exactly the coefficient that appears in the expansion of $(x+y)^n$, where each term corresponds to the number of ways to select a certain number of objects with and without specific properties.

4.4.2.3 Applications in Probability

The Binomial Theorem also plays a key role in probability theory, especially in problems involving the binomial distribution. The binomial distribution describes the number of successes in n independent Bernoulli trials, each with a probability p of success. The probability of exactly k successes is given by:

$$P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

This formula arises from the binomial expansion, where the terms represent the different ways that k successes can occur in n trials, with each trial having a success probability p.

4.4.3 Example: Counting Combinations with the Binomial Theorem

Suppose we have a set of 5 objects: A, B, C, D, E. How many ways can we choose 2 objects from this set?

Using the Binomial Theorem, we recognize that the number of ways to choose 2 objects is given by $\binom{5}{2}$. From the Binomial formula:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \times 4}{2 \times 1} = 10.$$

Thus, there are 10 ways to choose 2 objects from the set $\{A, B, C, D, E\}$.

4.5 Introduction to Graph Theory

Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph consists of a set of vertices (also called nodes) connected by edges (also called arcs or links), and it provides a powerful framework for analyzing the relationships between different objects. Graph theory is foundational in many areas of mathematics, computer science, engineering, and applied sciences, such as social networks, transportation systems, and communication networks.

This section introduces basic graph theory concepts and explores their applications in solving network-related problems. We will cover foundational defns, various types of graphs, important theorems, and the real-world applications of graph theory.

4.5.1 Basic defns and Concepts

Definition

A **graph** G is an ordered pair $G = (V, E)$, where:

- V is the set of vertices (or nodes), which represent the objects or entities in the network.
- E is the set of edges, which represent the connections between pairs of vertices. Each edge connects two vertices and can be either directed or undirected.

Graphs can be used to represent networks such as social networks, computer networks, road maps, and biological systems. A graph is often visualized as a set of points (vertices) connected by lines (edges).

Example

Consider a graph with vertices $V = \{A, B, C, D\}$ and edges $E = \{\{A, B\}, \{A, C\}, \{B, C\}, \{B, D\}, \{C, D\}\}$. This graph represents a network where the vertices are connected by edges, and each edge represents a relationship between two vertices.

Definition

The **degree** of a vertex is the number of edges incident to that vertex. The degree indicates how many connections a vertex has within the graph.

Example

In the graph from the previous example, the degree of vertex B is 3, because there are 3 edges connected to B: one connecting to A, one to C, and one to D.

The degree of a vertex helps us understand its centrality and connectivity within the network. Vertices with higher degrees are often considered more influential or central in various network analyses.

4.5.2 Types of Graphs

Graphs can be classified into different types based on their properties. Here are some of the most commonly studied types of graphs:

- **Directed Graph (Digraph)**: In a directed graph, each edge has a direction, meaning that the edges are ordered pairs. For example, an edge $\{A, B\}$ represents a directed relationship from A to B.
- **Undirected Graph**: In an undirected graph, the edges do not have direction, meaning that the relationship between the vertices is bidirectional. An edge $\{A, B\}$ represents a connection between A and B without a specific direction.
- **Weighted Graph**: In a weighted graph, each edge has a weight or cost associated with it, often representing distance, time, or any other quantity that can be quantified.
- **Complete Graph**: A complete graph is a graph in which every pair of distinct vertices is connected by a unique edge.
- **Bipartite Graph**: A bipartite graph is one in which the vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent. Bipartite graphs are often used in modeling relationships between two different classes of objects.
- **Tree**: A tree is a special type of graph that is connected and acyclic. Trees have important applications in hierarchical data structures, such as file systems and decision trees.
- **Planar Graph**: A planar graph is one that can be drawn on a plane without any edges crossing each other. The study of planar graphs is central to topology and circuit design.

Each type of graph has its own set of properties and applications, making them suitable for modeling different kinds of systems.

4.5.3 Historical Development of Graph Theory

The origins of graph theory can be traced back to the 18th century when the Swiss mathematician **Leonhard Euler** solved the famous *Seven Bridges of Königsberg* problem in 1736. The problem was based on the city of Königsberg (now Kaliningrad, Russia), which was divided by a river and had seven bridges connecting different parts of the city. The question was whether it was possible to walk through the city and cross each bridge exactly once without retracing any steps.

Euler's groundbreaking solution involved representing the landmasses and bridges as vertices and edges, respectively, of a graph. He concluded that it was impossible to traverse all the bridges exactly once, as the graph did not meet certain conditions. This work is considered the birth of graph theory, and Euler's *Eulerian Path Theorem* laid the foundation for the study of paths in graphs.

Historical Fact

Euler's solution to the Seven Bridges of Königsberg problem marked the birth of graph theory as a distinct mathematical field. This problem laid the foundation for the study of graph traversal, connectivity, and pathfinding.

Since Euler's time, graph theory has expanded dramatically, with contributions from mathematicians such as **Carl Friedrich Gauss**, **Kurt Gödel**, and **Paul Erdős**. Over the years, graph theory has found numerous applications in computer science, biology, economics, and social science, from optimizing network flows to understanding the structure of social networks.

4.5.4 Important Theorems in Graph Theory

Graph theory has numerous important theorems that form the backbone of the field. Here are a few key ones:

Theorem 5 [title=Euler's Formula]. For any connected planar graph, the following equation holds:

$$V - E + F = 2,$$

where V is the number of vertices, E is the number of edges, and F is the number of faces (regions) in the graph. This theorem is fundamental to the study of planar graphs.

Theorem 6 [title=Handshaking Lemma]. In any undirected graph, the sum of the degrees of all vertices is twice the number of edges. That is:

$$\sum_{v \in V} \deg(v) = 2E,$$

where $\deg(v)$ is the degree of vertex v, and E is the number of edges in the graph.

Theorem 7 [title=Graph Coloring Theorem]. For any planar graph, the minimum number of colors required to color the vertices of the graph such that no two adjacent vertices have the same color is at most 4. This is known as the *Four Color Theorem*.

7.0.1 Applications of Graph Theory

Graph theory has wide-ranging applications in various fields. Some of the key applications include:

- **Computer Networks**: Graph theory helps in the design and optimization of routing algorithms, such as the shortest path algorithm (e.g., Dijkstra's algorithm) and network flow algorithms (e.g., Ford-Fulkerson algorithm). These algorithms are used in the design of efficient computer networks, such as the internet.
- **Social Networks**: In social network analysis, graphs represent individuals as vertices and relationships as edges. Graph theory is used to study social dynamics, identify influential individuals, and detect communities within networks.
- **Transportation Systems**: Graphs are used to model transportation networks, such as road networks, flight routes, and rail systems. Graph theory helps optimize routing, minimize travel time, and improve overall system efficiency.

- **Biology**: In biological research, graph theory is used to model protein-protein interaction networks, gene regulatory networks, and metabolic pathways. These networks help understand cellular processes and disease mechanisms.
- **Operations Research**: Graphs are widely used in operations research to model transportation and supply chain networks, helping to minimize costs and maximize efficiency.

By providing a mathematical framework for understanding complex relationships and interactions, graph theory plays a crucial role in solving real-world problems in many domains.

7.1 Advanced Topics in Combinatorics and Graph Theory

7.1.1 Inclusion-Exclusion Principle

The **Inclusion-Exclusion Principle** is a powerful combinatorial method used to calculate the cardinality of the union of multiple sets, particularly when the sets may overlap. This principle allows us to account for the overcounting that occurs when sets have common elements.

Definition

The Inclusion-Exclusion Principle for two sets A and B states:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For three sets A, B, and C, the principle becomes:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

This principle extends to any number of sets. The key idea is to sum the sizes of the individual sets, subtract the sizes of their pairwise intersections (since those elements were counted twice), and then add back the intersection of all three sets (as it was subtracted too many times).

Example

Consider a problem where we want to find the number of people who have either a pet dog, a pet cat, or both. Let A represent people with a dog, and B represent people with a cat. If we know the sizes of $|A|$, $|B|$, and $|A \cap B|$ (the number of people with both a dog and a cat), we can apply the Inclusion-Exclusion Principle to calculate the total number of people with a pet dog or cat.

The Inclusion-Exclusion Principle is widely used in problems involving counting with restrictions, probability, and partitioning.

7.2 Easy Problems

7.2.1 Combinatorics Problems

1. **Counting Arrangements:** How many ways can you arrange the letters in the word **APPLE**?
2. **Choosing Items:** How many ways can you select 2 items from a set of 5 items?
3. **Permutations of Numbers:** How many ways can you arrange the digits of the number 12345?
4. **Selecting Books:** From a collection of 6 different books, how many ways can you select 3 books to read?
5. **Arrangements with Restrictions:** How many ways can you arrange 4 boys and 3 girls in a row if the boys must be placed together?
6. **Selecting Different Items:** How many ways can you select 4 out of 7 items if the order of selection does not matter?
7. **Simple Graph Counting:** How many edges are there in a complete graph with 4 vertices?
8. **Subset Selection:** How many subsets can you form from the set $S = \{1, 2, 3\}$?
9. **Simple Counting of Paths:** In a 2x2 grid, how many different paths are there from the bottom-left corner to the top-right corner, moving only up or to the right?
10. **Choosing Elements with Repetition:** How many ways can you choose 3 apples from a basket containing 5 apples, where you can choose any number of apples, even the same one, each time?
11. **Birthday Paradox:** What is the probability that in a room of 23 people, at least two of them share the same birthday?
12. **Simple Graph Problem:** How many edges are in a complete graph with 5 vertices?
13. **Choosing a Committee:** How many ways can you form a committee of 3 people from a group of 8 people?
14. **Coin Tosses:** How many ways can you get exactly 2 heads when tossing 3 coins?
15. **Distinct Numbers:** How many ways can you arrange the numbers 1, 2, and 3 in a sequence?

7.2.2 Graph Theory Problems

1. **Basic Graph defn:** Draw a simple graph with 3 vertices and 2 edges.
2. **Paths in Graphs:** In a graph with 4 vertices, how many simple paths (paths without repeating vertices) are there from vertex A to vertex D if there are direct edges between each adjacent vertex?
3. **Degree of a Vertex:** In the graph $G = \{\{A, B\}, \{B, C\}, \{C, D\}, \{A, C\}\}$, what is the degree of vertex B?

4. **Counting Edges in a Simple Graph:** How many edges are in a graph with 5 vertices and 4 edges?

5. **Complete Graph:** How many edges are in a complete graph with 6 vertices?

6. **Cycle in a Graph:** Can a graph with 4 vertices and 3 edges form a cycle? Draw an example.

7. **Connected Graph:** Is the graph $G = \{\{A, B\}, \{B, C\}, \{D, E\}\}$ connected? Why or why not?

8. **Graph Types:** Draw a bipartite graph with 4 vertices, 2 vertices in each set.

9. **Graph with No Loops:** Can a graph with 5 vertices and 7 edges have any loops? Draw a possible graph.

10. **Paths and Cycles:** In a cycle of 4 vertices, what is the maximum number of edges that can be removed while still keeping the graph connected?

7.3 Medium Problems

1. **Partitions of a Set:** How many ways can you partition the set $S = \{1, 2, 3, 4\}$ into two non-empty subsets? *Source: Aigner, M., and Ziegler, G. (2004). Proofs from THE BOOK. Springer-Verlag.*

2. **Paths in a Grid:** Consider a grid where you can only move either up or to the right at each step. How many distinct paths are there from the bottom-left corner to the top-right corner of a 3x3 grid? *Source: Knuth, D. E. (1997). The Art of Computer Programming, Vol 1: Fundamental Algorithms. Addison-Wesley.*

3. **Graph Connectivity:** In a simple graph with 6 vertices, each vertex has exactly 3 edges connected to it. How many edges are in this graph? *Source: Guy, R. K. (2003). Unsolved Problems in Combinatorial Mathematics. Springer.*

7.4 Hard Problems

1. **Counting Combinations with Restrictions:** How many ways can you select 4 students from a group of 10, given that two specific students must either both be selected or both be excluded?

2. **Graph Coloring:** In a graph with 6 vertices, where each vertex is connected to every other vertex (a complete graph), what is the minimum number of colors needed to color the vertices so that no two adjacent vertices have the same color? *Source: Aigner, M., and Ziegler, G. (2004). Proofs from THE BOOK. Springer-Verlag.*

3. **Olympiad-Style Problem: Number of Hamiltonian Cycles in a Complete Graph:** How many distinct Hamiltonian cycles are there in the complete graph on 6 vertices? *Source: Knuth, D. E. (1997). The Art of Computer Programming, Vol 1: Fundamental Algorithms. Addison-Wesley.*

4. **Graph Theoretic Proof:** Prove that in any simple graph with n vertices, if the number of edges is greater than or equal to $\frac{n(n-1)}{4}$, the graph must contain a triangle (a cycle of length 3). *Source: Guy, R. K. (2003). Unsolved Problems in Combinatorial Mathematics. Springer.*